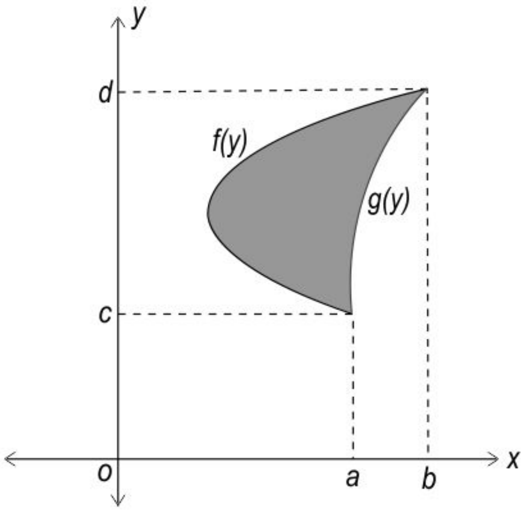
Application of Integrals

Multiple Choice Questions

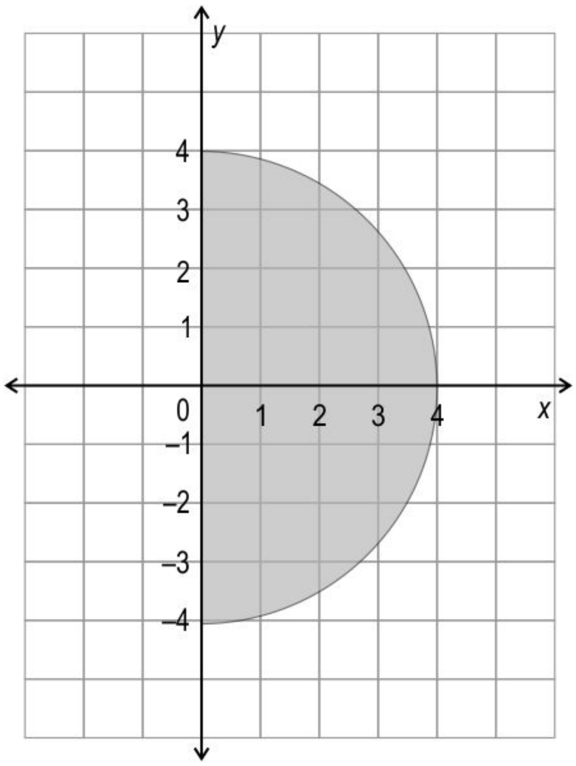
Q: 1 **Shown below are partial graphs of two distinct functions, *f* (*y*) and *g* (*y*). The region between them is shaded.**



Which expression gives the area of this shaded region?

1. $\int_a^b |f(y) - g(y)|\, dy$
2. $\int_a^b |f(x) - g(x)|\, dx$
3. $\int_c^d |f(y) - g(y)|\, dy$
4. $\int_c^d |f(x) + g(x)|\, dx$

Q: 2 **Shown below is the graph of the function $x^2 + y^2 = 16$, $0 \le x \le 4$.**



Which of the following expressions represents the area of the shaded region?

1 $2\int_0^4 \sqrt{16-y^2}dy$

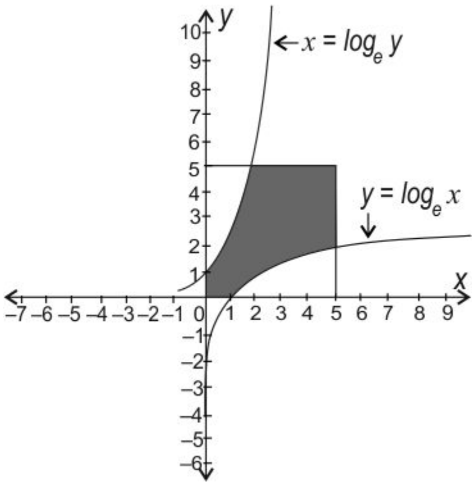
2 $\left|\int_{-4}^{0} (16-y^2)\, dy\right| + \int_0^4 (16-y^2)\, dy$

3 $\left|\int_{-4}^{0} \sqrt{16-x^2}\, dx\right| + \int_0^4 \sqrt{16-x^2}\, dx$

4 $\int_0^4 \sqrt{16-x^2}\, dx$

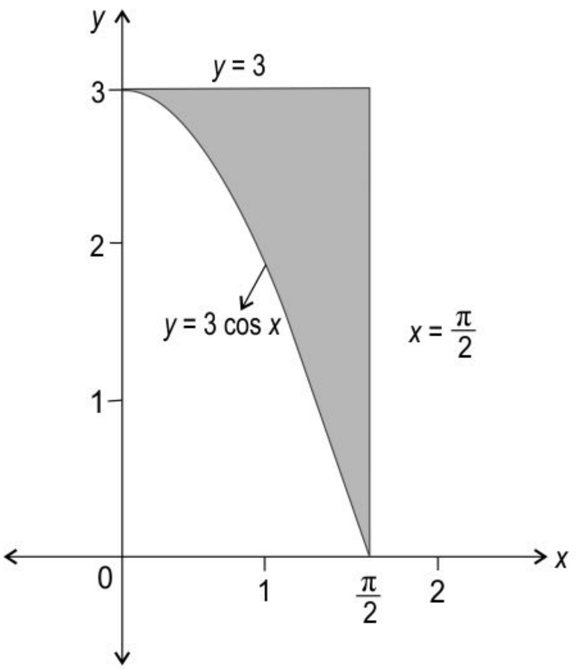
Free Response Questions

Q: 3 **Look at the graph below.** **[1]**



Write an expression for the area of the shaded region.

Q: 4 **Find the area of the shaded region in the graph shown below.** **[1]**



Show your work.

Q: 5 The area bounded by the lines 2 y = x - 2, x = 1 and x = 4 can be found as follows: [1]

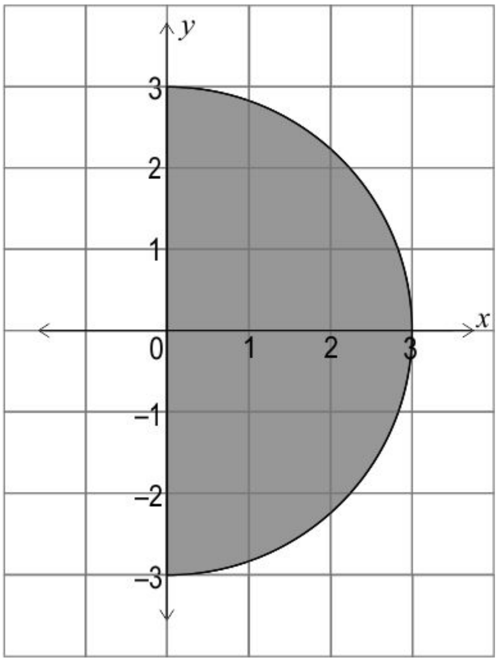
$$\int_1^4 \left(\frac{1}{2}x - 1\right)dx$$

State true or false. If true, justify your answer. If false, write the correct expression for the area.

Q: 6 Roshan wants to calculate the area bounded by the curve $x^2 + |y| = 4$. [1]

Write an expression that can be used to calculate the area correctly. Justify your answer.

Q: 7 Shown below is the graph of the function $x^2 + y^2 = 9$, $0 \le x \le 3$. [1]



Rajiv, Swara and Zaman represented the area of the shaded region in the following ways:

Rajiv: $\int_{-3}^{3} \sqrt{9 - x^2}dx$

Swara: $2\int_0^3 \sqrt{9 - y^2}dy$

Zain: $\left|\int_{-3}^{0} \sqrt{9 - x^2}dx\right| + \int_0^3 \sqrt{9 - x^2}dx$

Who represented it correctly and who did not? Justify your answer with a valid reason in each case.

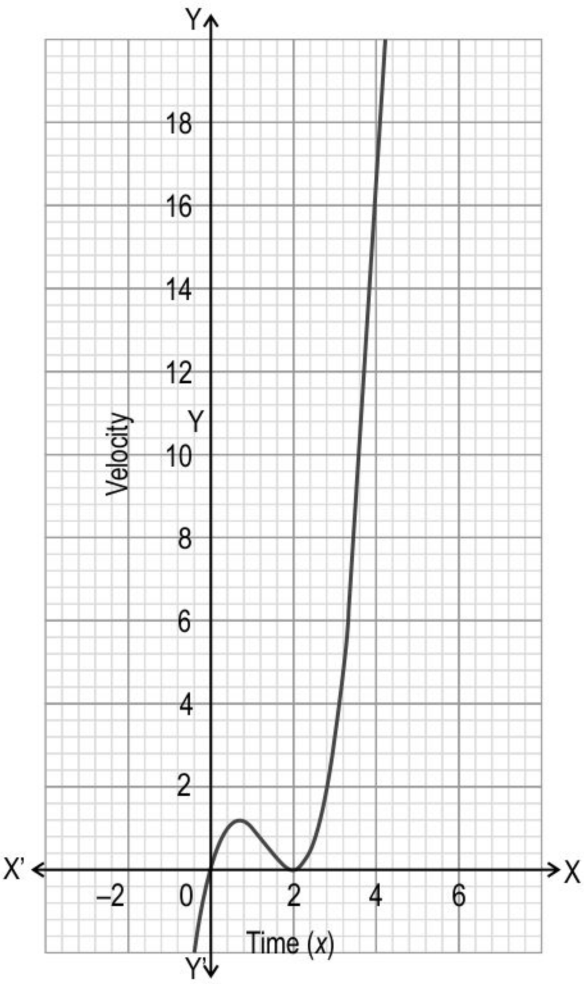
Q: 8 **The velocity function of an object is $v(t) = t^2 + 4t - 5$, where t is in hours and $v(t)$ is in kilometers per hour.** **[2]**

Find the displacement of the object during the first 3 hours. Show your steps.

Q: 9 **The area of the region bounded by $y = 4x^3 - kx^2 + 1$ and the x-axis, between the lines $x = 0$ and $x = 2$ is 10 sq units, where $k \in R$.** **[2]**

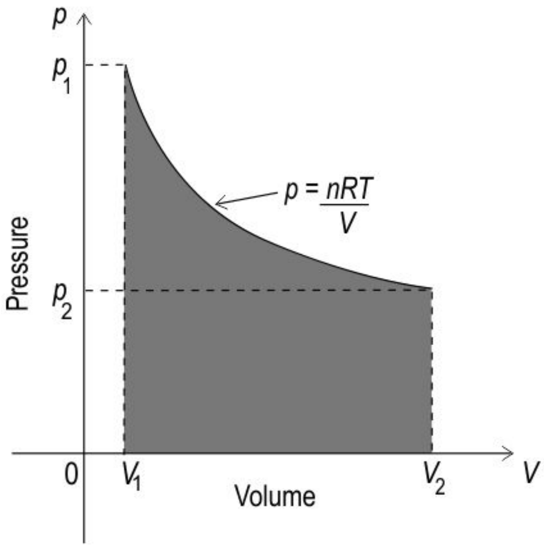
Find the value of k. Show your steps.

Q: 10 **The velocity of a particle, $v(x)$ is given by $(x - 2)^3 + 2(x - 2)^2$ m/s, where x is the time, as shown in the graph below.** **[2]**



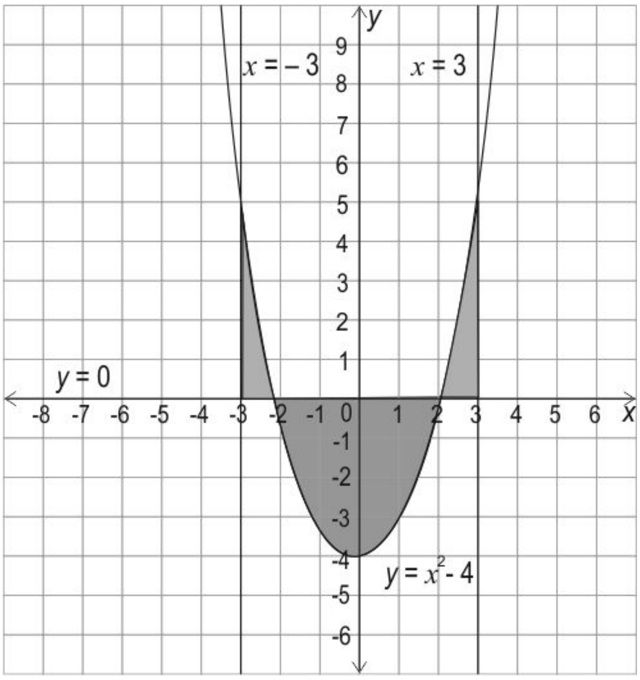
Find the displacement of the particle from 0 to 4 seconds. Show your steps.

Q: 11 **As n moles of a gas undergoes an expansion under constant temperature T, its volume V increases and it does some work on its surroundings. This work is represented by the area under the curve shown in the pressure-volume (pV) diagram below:** **[2]**



Find the expression for the amount of work done by the gas if $pV = nRT$, where R is the gas constant. Show your work.

Q: 12 **The shaded region shown in the graph below represents the area bounded by the graph of $f(x) = x^2 - 4$, the lines $x = -3$, $x = 3$ and the x -axis.** **[5]**



Find the area of the shaded region. Show your steps.

Q: 13 **A certain region is bounded by the x -axis and the graph of $y = \sin \frac{x}{2}$ between $x = 0$ and $x = 2\pi$. It is then divided into 2 regions by the line $x = k$.** **[5]**

If the area of the region between $x = 0$ and $x = k$ is thrice the area of the region between $x = k$ and $x = 2\pi$, find k. Draw a rough figure and show your steps.

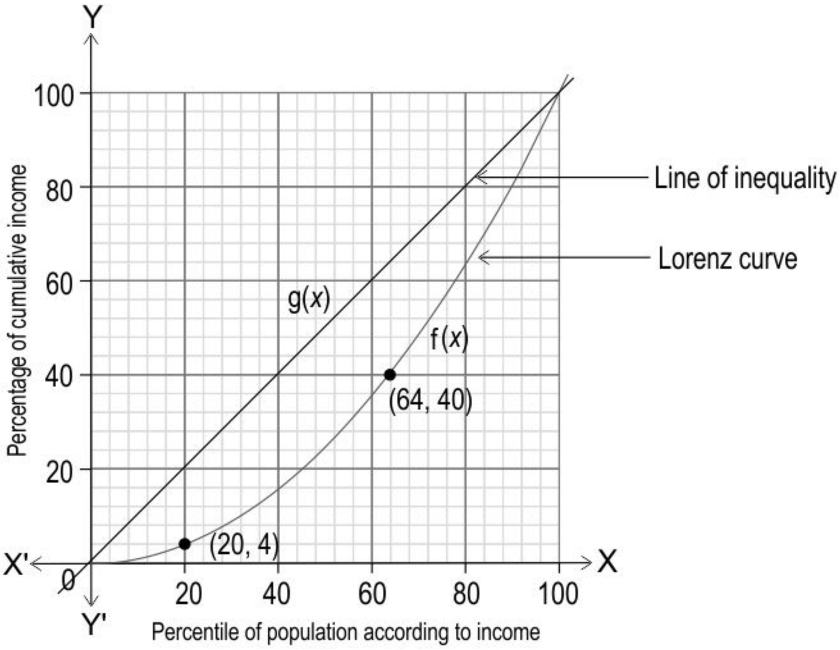
Case Study

Answer the following questions based on the given information.

A Lorenz curve is used to graphically represent income inequality in a society. It was developed by Max Lorenz in 1905.

In this curve, the percentile of the population according to their income is plotted on the x axis, and on the y axis, the percentage of cumulative income from that percentile of the population is plotted.For example, in the graph below, the point (20, 4) denotes that people with more income than that of 20% of the total population contribute 4% of the total income for the country. Similarly, the point (64, 40) denotes that people with more income than that of 64% of the country's population contribute 40% of the total income for the country.
On the graph, there is also a line of equality, given by the function g(x) = x. The further away the Lorenz curve of a society is from the line of equality, the more unequal its income distribution is.

In order to compare this data for multiple countries, the area under the Lorenz Curve is used to find the Gini coefficient (G), whose value ranges between 0 and 1. The closer G's value is to 1, the more unequal the income distribution of the society is.



The Lorenz curve shown above can be approximated by the following function:

$$f(x) = \frac{\sqrt{x}}{100} + \frac{(x-1)^2}{100}$$

Q: 14 **Find the area under the Lorenz Curve from 0 to 100. Show your work.** **[3]**

Q: 15 **The Gini coefficient is given by G = $\frac{A}{A+B}$, where A is the area between the Lorenz curve and the line of equality, and B is the area under the Lorenz curve.** **[2]**

Find the Gini coefficient for the given Lorenz curve, using integration. Show your work.

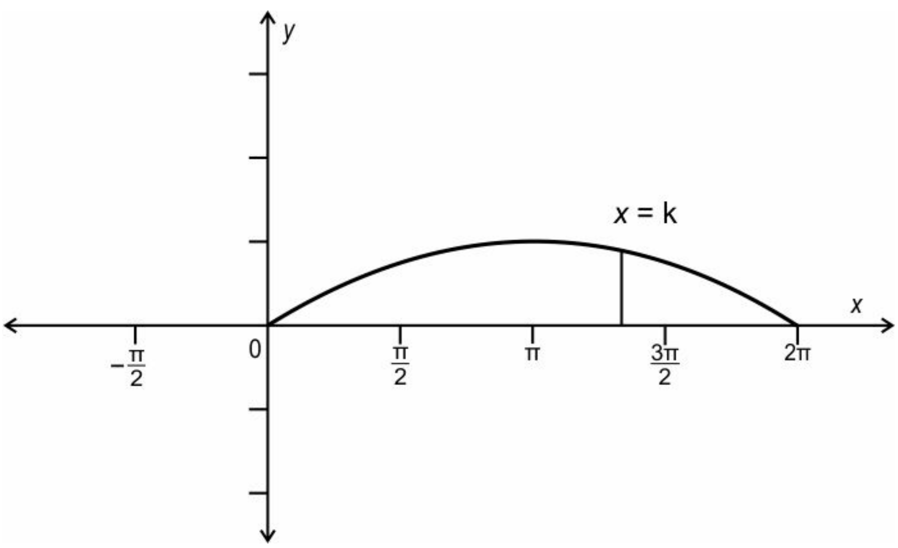
The table below gives the correct answer for each multiple-choice question in this test.

Q.No	Correct Answers
1	3
2	1

Q.No	What to look for	Marks
3	**Writes an expression for the area of the shaded region as:** Shaded area = $25 - 2\int_1^5 \log_e x \, dx$ **(Award full marks for any equivalent expression.)**	**1**
4	**Represents the area of the shaded region as:** $\frac{3\pi}{2} - \int_0^{\frac{\pi}{2}} 3\cos x dx$	**0.5**
	Simplifies the above expression to find the area of the shaded region as follows: $\frac{3\pi}{2} - 3\,[\sin x]_0^{\frac{\pi}{2}}$ $= 3\left[\frac{\pi}{2} - 1\right]$	**0.5**
5	**Writes false.**	**0.5**
	States that the line 2 $y = x$ - 2 crosses the x -axis at $x = 2$ and hence the area under the curve will have to be found as follows: $\left\vert\int_1^2 (\frac{1}{2}x - 1)dx\right\vert + \int_2^4 (\frac{1}{2}x - 1)dx$	**0.5**
6	**Writes that the area is equivalent to the area bounded by $x^2 + y = 4$ and $x^2 - y = 4$.**	**0.5**
	Writes the following expression to calculate the bounded area: $\int_{-2}^{2} (4 - x^2)\ dx + \left\vert\int_{-2}^{2} (x^2 - 4)\ dx\right\vert$ **(Award full marks for any equivalent expression.)**	**0.5**

Q.No	What to look for	Marks
7	**Writes that only Swara represented the area of the shaded region correctly. Justifies the answer. For example, the parts of shaded region above and below the *x* -axis are equal. Hence the required area can be computed by integrating the function with respect to *y* -axis from 0 to 3 and then doubling it.**	**0.5**
	Writes that Rajiv and Zaman represented the area of the shaded region incorrectly. Justifies the answer. For example, if integrated along *x* -axis, the limit must range from 0 to 3. But Rajiv and Zaman have integrated from (-3) to 3.	**0.5**
8	**Writes the expression for the displacement of the object using the velocity function as follows:** $\int_0^3 (t^2 + 4t - 5)dt$	**0.5**
	Evaluates the above definite integral as: $\left[\frac{t^3}{3} + \frac{4t^2}{2} - 5t\right]_0^3$	**1**
	Applies the given limit to find the displacement as 9 + 18 - 15 = 12 km.	**0.5**
9	**Sets the integral equation as:** $\int_0^2 (4x^3 - kx^2 + 1)dx = 10$	**0.5**
	Simplifies the above equation as: $\left[\frac{4x^4}{4} - \frac{kx^3}{3} + x\right]_0^2 = 10$	**0.5**
	Applies the limit and simplifies the above equation as: $16 - \frac{8k}{3} + 2 = 10$	**0.5**

Q.No	What to look for	Marks
	Simplifies the above equation to get *k* as 3.	**0.5**
10	Integrates $\int_0^4((x-2)^3+2(x-2)^2)dx$ as $\left[\frac{(x-2)^4}{4}\right]_0^4+\left[\frac{2(x-2)^3}{3}\right]_0^4$	**1**
	Evaluates the above integral as $\frac{32}{3}$ and hence finds the displacement of the particle from 0 to 4 seconds as $\frac{32}{3}$ m.	**1**
11	**Represents the work done by the gas as:** $\int_{V_1}^{V_2} pdV$ $= nRT\int_{V_1}^{V_2}\frac{1}{V}dV$	**1**
	Integrates the above expression to find the work done by the gas as: $nRT\,[\log V]_{V_1}^{V_2}$ $= nRT\,[\log V_2 - \log V_1]$ $= nRT\log\left(\frac{V_2}{V_1}\right)$	**1**
12	**Sets up the integrals as follows:** $\text{Area} = \int_{-3}^{-2}(x^2-4)dx + \left\|\int_{-2}^{2}(x^2-4)dx\right\| + \int_2^3(x^2-4)dx$	**1**
	Evaluates the area from *x* = -3 to *x* = -2 as follows: $\int_{-3}^{-2}(x^2-4)dx = \left[\frac{x^3}{3}-4x\right]_{-3}^{-2} = \frac{7}{3}$	**1**

Q.No	What to look for	Marks
	Evaluates the area from x = -2 to x = 2 as follows: $\left\lvert\int_{-2}^{2}(x^2-4)dx\right\rvert = \left\lvert\left[\frac{x^3}{3}-4x\right]_{-2}^{2}\right\rvert = \left\lvert\frac{-32}{3}\right\rvert = \frac{32}{3}$	**1.5**
	Evaluates the area from x = 2 to x = 3 as follows: $\int_{2}^{3}(x^2-4)dx = \left[\frac{x^3}{3}-4x\right]_{2}^{3} = \frac{7}{3}$	**1**
	Adds the area found in the above 3 steps to find the area of the shaded region as: $\frac{7}{3}+\frac{32}{3}+\frac{7}{3}=\frac{46}{3}$ or 15.33 sq units	**0.5**
13	**Draws a rough graph of sin $\frac{\pi}{2}$ and x = k.**  **(Award only 0.5 marks if the range on the x -axis is not from 0 to 2π.)**	**1**

Q.No	What to look for	Marks
	Writes the equation for the given situation as: $\int_0^k sin\frac{x}{2}dx = 3\int_k^{2\pi} sin\frac{x}{2}dx$	**1**
	Simplifies the above equation as: $\left[-2\ cos\frac{x}{2}\right]_0^k = 3\left[-2\ cos\frac{x}{2}\right]_k^{2\pi}$	**1**
	Simplifies the above equation as: $-2\cos\frac{k}{2} + 2\cos 0 = -6\cos\pi + 6\cos\frac{k}{2}$	**1**
	Simplifies the above equation to get *k* as: $\frac{k}{2} = \cos^{-1}\frac{-1}{2} = \frac{2\pi}{3}$ $=> k = \frac{4\pi}{3}$	**1**
14	**Rewrites the integral of f(*x*) as follows:** $\int_0^{100}(\frac{\sqrt{x}}{100} + \frac{(x-1)^2}{100})dx$ $= \int_0^{100}\frac{\sqrt{x}}{100}dx + \int_0^{100}\frac{(x-1)^2}{100}dx$	**0.5**
	Solves $\int_0^{100}\frac{\sqrt{x}}{100}dx = \left[\frac{2x^{\frac{3}{2}}}{300}\right]_0^{100}$ **Evaluates this to find the answer as $\frac{20}{3}$.** **(Award only 0.5 marks for correctly solving the integration, but incorrectly evaluating the expression.)**	**1**

Q.No	What to look for	Marks
	Solves $\int_0^{100} \frac{(x^2-2x+1)}{100} dx = \left[\frac{x^3}{300}\right]_0^{100} - \left[\frac{2x^2}{200}\right]_0^{100} + \left[\frac{x}{100}\right]_0^{100}$ **Evaluates this to find the answer as $\frac{10000}{3}$ - 100 + 1 = $\frac{10000}{3}$ - 99.** **(Award only 0.5 marks for correctly solving the integration, but incorrectly evaluating the expression.)**	**1**
	Adds the values from steps 2 and 3 to get: **$\frac{20}{3}$ + $\frac{10000}{3}$ - 99 = 3241 sq units.**	**0.5**
15	**To find A + B, integrates the area under g(x) = x as 5000 sq units.** **The integration may look as follows:** $\int_0^{100} x dx = \left[\frac{x^2}{2}\right]_0^{100} = 5000$	**1**
	Finds A as (5000 - 3241) = 1759 sq. units.	**0.5**
	Calculates G = 1759/5000 = approximately 0.35.	**0.5**